Integer solutions of $n^m = m^n$

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1 Introduction

This document is a mathematical proof of the following statement:

> *16 is the only number that equals n^m and m^n for two unique positive integers n and m.*

2 Constraints and Assumptions

Here are the constraints for this problem:

$$n, m \in \mathbf{N} \tag{1}$$

$$n \neq m \tag{2}$$

$$n^m = m^n \tag{3}$$

And based on (2), we can make this assumption:

$$m < n \tag{4}$$

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3 Proof

Based on (2) and (4), we can deduce that

$$\exists\ k \in \mathbf{N} \mid m + k = n \tag{5}$$

Substituting (5) in (3):

$$\begin{aligned} (m+k)^m &= m^{m+k} \\ \implies m^m \cdot \left(1 + \frac{k}{m}\right)^m &= m^m \cdot m^k \\ \implies \left(1 + \frac{k}{m}\right)^m &= m^k \qquad (6) \\ \because m^k \in \mathbb{Z} \quad &\therefore \frac{k}{m} \in \mathbb{Z} \end{aligned}$$

Thus, we know that m divides k.

$$\exists\ p \in \mathbb{Z} \mid k = p \cdot m \tag{7}$$

Substituting (7) in (5), we get:

$$\begin{aligned} &n = m + p \cdot m \\ &n = (1+p) \cdot m \qquad (8) \end{aligned}$$

Substituting (8) back into (6), we get:

$$\begin{aligned} (1+p)^m &= m^{pm} \\ \because m \in \mathbf{N}\ \therefore m &\neq 0 \\ \implies 1 + p &= m^p \\ p &= m^p - 1 \\ p &= (m-1)(1 + m + m^2 + \cdots + m^{p-1}) \qquad (9) \end{aligned}$$

Refer to Appendix A for derivation of the expansion.
Now, again, because $m \in \mathbf{N}$, thus the minimum value of m is 1. Because there are p terms in the expansion:

$$1 + m + m^2 + \cdots + m^{p-1} \geq p \tag{10}$$

Substituting (10) into (9), we get:

$$p = (m-1)(1 + m + m^2 + \cdots + m^{p-1}) \geq (m-1)p$$
$$p \geq (m-1)p \tag{11}$$

We know that $p \neq 0$, because $p = 0 \implies n = m$ which contradicts our constraint (2): $n \neq m$. Therefore,

$$1 \geq m - 1$$
$$\therefore m \leq 2$$
$$eq.(1) \implies m \in \{1, 2\}$$

If $m = 1$, equation (3) becomes:

$$n^1 = 1^n$$
$$n = 1 \ = m$$
$$\therefore m = n \nleftrightarrow eq.(2)$$

Thus, only one value for m qualifies.

$$\boxed{m = 2} \tag{12}$$

Substituting (12) to (3):

$$n^2 = 2^n$$

Since 2 is a prime number, therefore the prime factorization of RHS only contains 2. Also, given that $n \neq m$, we know that $n \neq 2$. Thus,

$$\exists\ a \in \mathbf{N} \mid n = 2^{a+1} \tag{13}$$
$$\implies (2^{a+1})^2 = 2^{2^{a+1}}$$
$$2^{2(a+1)} = 2^{2^{a+1}}$$
$$2 \cdot (a+1) = 2^{a+1}$$
$$a + 1 = 2^a$$
$$a = 2^a - 1$$
$$a = (2-1)(1 + 2 + 2^2 + \cdots + 2^{a-1})$$
$$a = 1 + 2 + 2^2 + \cdots + 2^{a-1}$$

$$\because a \in \mathbf{N}$$
$$\therefore 1 + 2 + 2^2 + \cdots + 2^{a-1} \geq a$$
$$\implies a = 2^a - 1 \geq a \tag{14}$$

The only way the equality (14) holds is with the minimum value of a, which is 1. Substituting $a = 1$ in equation (13), we get:

$$n = 2^{1+1} = 2^2 \implies \boxed{n = 4} \tag{15}$$

Based on (12) and (15), we can conclude that:

$$m = 2; n = 4$$
$$\boxed{m^n = n^m = 16}$$

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A Expansion of $m^p - 1$

This is the proof of the following expansion used multiple times in the document:

$$m^p - 1 = (m-1)(1 + m + m^2 + \cdots + m^{p-1}) \tag{16}$$

A.1 Approach

We will start from the RHS of (16). One of its terms is a sum of an GP (Geometric Progression). We will evaluate the value of that GP. Here is the GP:

$$S = 1 + m + m^2 + \cdots + m^{p-1} \tag{17}$$

A.2 Proof

Multiplying both sides of the above equation (17) by the common multiplier m of the GP, we get:

$$m \cdot S = m + m^2 + m^3 + \cdots + m^{p-1} + m^p \tag{18}$$

Notice that the $p-1$ terms, which are m, m^2,...,m^{p-1}, are common in the RHS of equations (17) and (18). So if we subtract S from $m \cdot S$, we get:

$$\begin{aligned} S \cdot m - S &= m + m^2 + \cdots + m^{p-1} + m^p - 1 - m - m^2 - \cdots - m^{p-1} \\ (m-1) \cdot S &= m^p - 1 \end{aligned} \tag{19}$$

Substituting (17) back into (19), we get:

$$\boxed{(m-1)(1 + m + m^2 + \cdots + m^{p-1}) = m^p - 1} \tag{20}$$

(20) is same as (16) ■